MoM is implemented based on the mixed potential integral equation formulation with a rectangular, but nonuniform and nonfixed, mesh. The entire formulation can be extended to multilayer substrates in a straightforward way.

APPENDIX

The key equation used is Sommerfeld's identity [11], which expresses a spherical wave in terms of cylindrical ones. This identity is quoted below, although slightly modified to serve our purposes

$$\frac{e^{-jkr}}{r} = \frac{1}{2j}\int_{-\infty}^{\infty} \frac{H_0^{(2)}(\lambda\rho)e^{-jz\sqrt{K^2-\lambda^2}}}{\sqrt{K^2-\lambda^2}}\lambda\, d\lambda, \qquad z > 0 \quad \text{(A1)}$$

where the path of integration is along the real-axis but passes above the branch-points at $\lambda = \pm K$ so that the radiation condition be satisfied. By taking the static limit in (A1), $K \to 0$, and using the substitution $\lambda \to j\lambda$, (i.e. transforming the path of integration from the real- to the imaginary axis), the following Fourier-pair relations are readily established:

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-j\lambda u} K_0(\lambda\rho)\, d\lambda = \frac{1}{2\sqrt{u^2+\rho^2}} \quad \text{(A2)}$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-j\lambda u}[\lambda K_1(\lambda\rho)]\, d\lambda = \frac{\rho}{2\,{}^{3/2}\!\sqrt{u^2+\rho^2}}. \quad \text{(A3)}$$

The Fourier pairs (A2) and (A3) are then used together with Poisson's summation formula to convert summations (7) and (8) into their accelerated representations of (9) and (10), respectively.

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Analysis of Electromagnetic Boundary-Value Problems in Inhomogenous Media with the Method of Lines

Arnd Kornatz and Reinhold Pregla

***Abstract*—In this paper we will show how the method of lines can be generalized for the analysis of inhomogenous media. The inhomogeneity is completely arbitrary; the permittivity of the investigated structures may vary in all three coordinate directions. Subjects under investigation are isolated dielectric resonators, mictrostrip filters with dielectric and metallic lossy resonators, and planar capacities.**

I. INTRODUCTION

In arbitrary inhomogenous media, electromagnetic fields can only be calculated with numerical methods. Possible methods are mode-matching methods [1], finite element methods [2], or finite difference methods [3]. As long as the medium is structured in some way, the numerical analysis is partly substitutable by analytical calculations. A method that is based on this principle is the method of lines [4]. If the structure is invariant in one coordinate direction, the fields can be calculated analytically in this direction. In the other directions, the calculation is furthermore discrete. In comparison to the above-mentioned methods [1]–[3] this procedure needs less computational resources. Under the use of Cartesian coordinates, the method of lines can be employed to analyze all structures that consist of layers in which the material does not change in normal direction. Every structure can be separated in such layers so that the method of lines is an universal tool for the analysis of arbitrary microwave components. Simple examples for layered structures are microwave filters with dielectric or metallic resonators, planar capacitors, optical modulators, or couplers. In spite of the differences between these structures (e.g., used materials, boundary conditions, and ranges of application), they can be analyzed with the same theory.

II. THEORY

A. Electrodynamic Applications

For the analysis of inhomogenous layers it is assumed that the permittivity ε_r varies in x and y-direction, but not in z-direction. In this case the electromagnetic field can be derived from a vector potential $\boldsymbol{A}$. It is important that the potential has the same vector components as the gradient of the permittivity

$$\boldsymbol{A} = A_x \cdot \boldsymbol{e}_x + A_y \cdot \boldsymbol{e}_y. \quad (1)$$

Only this general solution leads to a consistent system of coupled differential equations for the potential components A_x and A_y

$$\varepsilon_r \frac{\partial}{\partial x}\left[\frac{1}{\varepsilon_r}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y}\right)\right] + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2} + k_0^2\varepsilon_r A_x = \frac{\partial}{\partial y}\left(\frac{\partial A_y}{\partial x}\right) \quad (2)$$

$$\frac{\partial^2 A_y}{\partial x^2} + \varepsilon_r \frac{\partial}{\partial y}\left[\frac{1}{\varepsilon_r}\left(\frac{\partial A_y}{\partial y} + \frac{\partial A_x}{\partial x}\right)\right] + \frac{\partial^2 A_y}{\partial z^2} + k_0^2\varepsilon_r A_y = \frac{\partial}{\partial x}\left(\frac{\partial A_x}{\partial y}\right). \quad (3)$$

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The authors are with the Allgemeine und Theoretische Elektrotechnik, Fern Universitaet Hagen, D-58084 Hagen, Germany.

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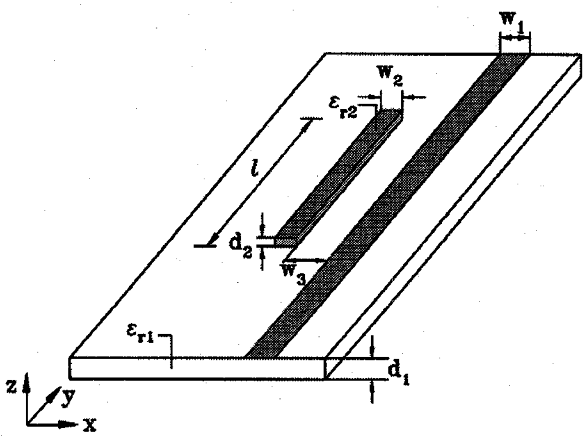

Fig. 1. Lossy metallic resonator excited by a microstrip line, $d_1 = 150\ \mu$m, $d_2 = 30\ \mu$m, $w_1 = 150\ \mu$m, $w_2 = 128.6\ \mu$m, $w_3 = 75\ \mu$m, $l = 2.55$ mm, $\varepsilon_{r1} = 12.9$, $\varepsilon_{r2} = -\mathrm{j}\ 1/\rho_{\mathrm{Cu}}\omega\varepsilon_0$.

In these self-adjoint differential equations all derivatives used are finite, even at dielectric wedges. So, this formulation is well suited for the numerical analysis. A similar solution was presented by [5] for one-dimensional (1-D) permittivity variations. The extension to two-dimensional (2-D) permittivity variations was first used for the investigation of integrated optical structures [6], [7]. When discretizing the differential equations, the differential operators $\partial/\partial x$ and $\partial/\partial y$ are substituted by difference operators. [8]. The discrete values of the potential are stored in the vector $\hat{\boldsymbol{A}}$. It is possible to derive the difference operators for the 2-D discretization from the ones for 1-D discretization using the Kronecker product [4]. The boundary conditions are directly considered in the difference operators. The discretization of a microwave filter as shown in Fig. 1 is described in [9] in detail. The potential of the incoming wave is $\hat{\boldsymbol{A}}_0$. An outgoing fundamental mode is assumed at the output of the filter. These boundaries lead to modified difference operators [9], [10]. After discretization a system of coupled ordinary differential equations remains

$$\frac{\partial^2}{\partial z^2}\hat{\boldsymbol{A}} - \hat{\boldsymbol{Q}}\hat{\boldsymbol{A}} = \hat{\boldsymbol{B}}\hat{\boldsymbol{A}}_0. \tag{4}$$

Transforming (4) to main axis gives a system of uncoupled differential equations

$$\frac{\partial^2}{\partial z^2}\hat{\overline{\boldsymbol{A}}} - \hat{\boldsymbol{k}}_z^2\hat{\overline{\boldsymbol{A}}} = \hat{\overline{\boldsymbol{B}}}\,\hat{\overline{\boldsymbol{A}}}_0 \tag{5}$$

with

$$\hat{\boldsymbol{k}}_z^2 = \hat{\boldsymbol{T}}^{-1}\hat{\boldsymbol{Q}}\,\hat{\boldsymbol{T}}; \qquad \hat{\overline{\boldsymbol{A}}} = \hat{\boldsymbol{T}}^{-1}\hat{\boldsymbol{A}}. \tag{6}$$

From the solution of (5) a relation between the potential and its derivatives with respect to z at the top and the bottom of a layer can be derived

$$\begin{bmatrix} \left.\frac{\partial}{\partial z}\hat{\overline{\boldsymbol{A}}}\right|_{z_1} \\ \left.\frac{\partial}{\partial z}\hat{\overline{\boldsymbol{A}}}\right|_{z_2} \end{bmatrix} = \hat{\overline{\boldsymbol{\Gamma}}}\begin{bmatrix} \hat{\overline{\boldsymbol{A}}}|_{z_1} \\ \hat{\overline{\boldsymbol{A}}}|_{z_2} \end{bmatrix} + \hat{\overline{\boldsymbol{\Gamma}}}_p\begin{bmatrix} \hat{\overline{\boldsymbol{A}}}_0|_{z_1} \\ \hat{\overline{\boldsymbol{A}}}_0|_{z_2} \end{bmatrix}. \tag{7}$$

This equation has to be transformed back to the spatial domain. With the discretized tangential electric and magnetic fields

$$\frac{k_0}{Z_0}\hat{\boldsymbol{E}}_t = \hat{\boldsymbol{p}}\hat{\boldsymbol{A}} + \hat{\boldsymbol{q}}\hat{\boldsymbol{A}}_0; \quad \hat{\boldsymbol{H}}_t = \frac{\partial}{\partial z}\hat{\boldsymbol{A}} \tag{8}$$

it is possible to eliminate the potentials $\hat{\boldsymbol{A}}$ and $\hat{\boldsymbol{A}}_0$ in (7). Now a relation between the tangential fields at the top and the bottom of a layer is obtained

$$Z_0\begin{bmatrix} \hat{\boldsymbol{H}}_t|_{z_1} \\ \hat{\boldsymbol{H}}_t|_{z_2} \end{bmatrix} = \hat{\boldsymbol{Y}}_{AB}\begin{bmatrix} \hat{\boldsymbol{E}}_t|_{z_1} \\ \hat{\boldsymbol{E}}_t|_{z_2} \end{bmatrix} + \hat{\boldsymbol{Y}}_{0AB}\begin{bmatrix} \hat{\boldsymbol{E}}_{0t}|_{z_1} \\ \hat{\boldsymbol{E}}_{0t}|_{z_2} \end{bmatrix}. \tag{9}$$

The linear equation system

$$\hat{\boldsymbol{Y}}\hat{\boldsymbol{E}}_{tF} + \hat{\boldsymbol{Y}}_0\hat{\boldsymbol{E}}_{0tF} = -Z_0\hat{\boldsymbol{J}}_F \tag{10}$$

is obtained by matching the fields at the interfaces between the layers. $\hat{\boldsymbol{E}}_{tF}$ contains the discretized tangential electric field, $\hat{\boldsymbol{E}}_{0tF}$ the exciting tangential electric field, and $\hat{\boldsymbol{J}}_F$ the discretized surface current at the interfaces. From (10) the electric field beside the metallization and the surface current on the metallization can be calculated.

It is also possible to investigate isolated resonators with the presented algorithm. The computational window has to be surrounded with absorbing boundaries [11]. Equation (10) changes to

$$\hat{\boldsymbol{Y}}(f)\hat{\boldsymbol{E}}_{tF} = 0. \tag{11}$$

The system matrix $\hat{\boldsymbol{Y}}$ is a function of the frequency f. The system equation has nontrivial solutions if $\det(\hat{\boldsymbol{Y}})$ disappears. Therefore, the resonance frequencies are the zeroes of $\det(\hat{\boldsymbol{Y}})$ [12].

B. Electrostatic Applications

In cartesian coordinates the differential equation for the scalar potential ϕ is

$$\frac{1}{\varepsilon_r}\left[\frac{\partial}{\partial x}\left(\varepsilon_r\frac{\partial\phi}{\partial x}\right)\right] + \frac{1}{\varepsilon_r}\left[\frac{\partial}{\partial y}\left(\varepsilon_r\frac{\partial\phi}{\partial y}\right)\right] + \frac{\partial^2\phi}{\partial z^2} = 0 \tag{12}$$

if the permeability ε_r is a function of x and y. If we discretize this equation in the same way as in the electrodynamic case we receive

$$\frac{\partial^2}{\partial z^2}\hat{\boldsymbol{\phi}} - \hat{\boldsymbol{Q}}\hat{\boldsymbol{\phi}} = 0. \tag{13}$$

To fulfill the interface conditions at dielectric discontinuities ε_r has to be calculated from the surrounding permittivities by

$$\frac{1}{\varepsilon_r} = \frac{\frac{1}{\varepsilon_{r1}} + \frac{1}{\varepsilon_{r2}}}{2}. \tag{14}$$

Transforming (13) to main axis gives a system of uncoupled differential equations

$$\frac{\partial^2}{\partial z^2}\hat{\overline{\boldsymbol{\phi}}} - \hat{\boldsymbol{k}}_z^2\hat{\overline{\boldsymbol{\phi}}} = 0. \tag{15}$$

From (15) we get the transmission line equations

$$\begin{bmatrix} \left.\frac{\partial}{\partial z}\hat{\overline{\boldsymbol{\phi}}}\right|_{z_1} \\ \left.\frac{\partial}{\partial z}\hat{\overline{\boldsymbol{\phi}}}\right|_{z_2} \end{bmatrix} = \hat{\overline{\boldsymbol{\Gamma}}}\begin{bmatrix} \hat{\overline{\boldsymbol{\phi}}}|_{z_1} \\ \hat{\overline{\boldsymbol{\phi}}}|_{z_2} \end{bmatrix} \tag{16}$$

between the top and the bottom of a layer. Matching the fields at the interfaces of all layers leads to the linear equation system

$$\hat{\boldsymbol{Y}}\hat{\boldsymbol{\phi}}_F = \boldsymbol{\varrho}_F. \tag{17}$$

This equation has to be separated into two equations. One equation determines the potential ϕ_F beside the metallization and the other equation determines the surface charge ρ_F on the metallization.

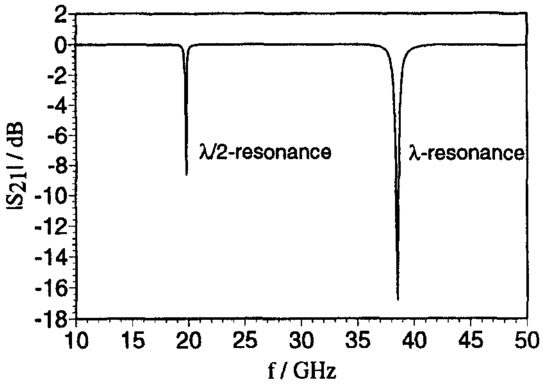

Fig. 2. $|S_{21}|$ of the lossy metallic resonator filter in Fig. 1 with 15×33 discretization lines for one component of the potential.

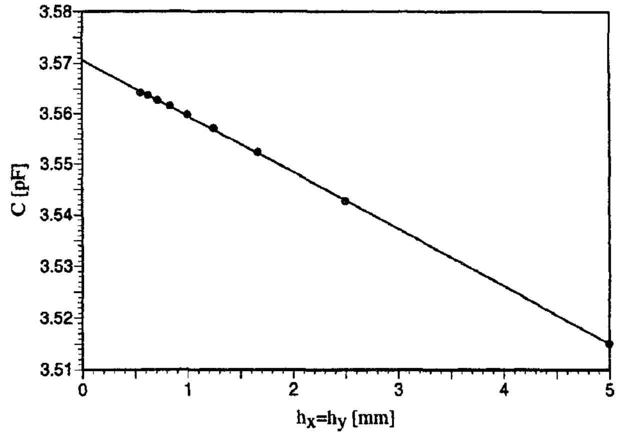

Fig. 3. Convergence of the capacitance as a function of the discretization distance, $\varepsilon_r = 38$, distance of the plates $d = 1$ cm, size of the quadratic plates $w = 1$ cm.

TABLE I
COMPARISON OF THE RESONANCE FREQUENCIES OF A CYLINDRICAL RESONATOR WITH THE RESULTS OF OTHER AUTHORS. $\varepsilon_r = 38$, HEIGHT = 4.6 mm, RADIUS $a = 5.25$ mm

Mode	$k_o a$ for resonance					
	our method	Theory [13]	Theory [14]	Theory [15]	Theory [16]	Measured [17]
$TE_{01\delta}$	0.537	0.537	0.531	0.534	0.535	0.533
$HEM_{11\delta}$	0.699	0.696	0.696	0.698	0.698	0.696
$HEM_{12\delta}$	0.731	0.732	0.730	0.731	0.731	0.726
$TM_{01\delta}$	0.825	0.827	0.827	0.829	0.827	0.824
$HEM_{21\delta}$	0.852	0.852	0.852	0.854	0.852	0.850

III. RESULTS

A metallic resonator with finite thickness that is excited by a microstrip line was chosen as an example for a dielectric resonator filter. Its finite conductance is modeled by a large imaginary permittivity. Fig. 1 shows the analyzed structure. Due to the calculation time very few discretization lines were used for the analysis. Fig. 2 shows the computed run of the transmission coefficient of the filter. As a second example an isolated dielectric resonator with circular shape was analyzed that is well known from other publications [13]–[17]. Its circular shape is modeled by a staircase function. We have calculated the field distributions, the resonance frequencies, and the quality factors due to radiation of the first five modes of the resonator. In Table I our results are compared with those of other authors. The resonance frequencies are in good correspondence. As an example for an electrostatic boundary problem, we have analyzed a plate capacitor which is filled with a dielectric material. Outside the capacitor a dielectric constant $\varepsilon_r = 1$ is assumed. Fig. 3 shows the computed capacitance C as a function of the discretization distance. The exact value can be obtained easily by an extrapolation of the computed values.

IV. CONCLUSION

We have shown how to analyze inhomogenous layered media by means of the full hybrid method of lines. This offers the possibility to investigate a wide range of high frequency devices like microwave filters, couplers, or resonators. The adaption to electrostatic problems allows to analyze planar capacitors. Examples are coupled gap, interdigital, dielectric overlay, or plate capacitors. Our results for isolated resonators show that the method of lines gives good results even if a small number of discretization lines is used. This has encouraged us to use the method of lines for the analysis of additional structures in the future.

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New Reciprocity Theorems for Chiral, Nonactive, and Biisotropic Media

Cesar Monzon

***Abstract*—Two generalized reciprocity theorems for homogeneous biisotropic media are presented that do not invoke a complementary space. One of them is eminently crosspolarized involving real sources and fields, while the other is a generalization of the Lorentz theorem and is therefore eminently copolarized, invoking generalized sources or fields. These theorems constitute the foundation for new variational expressions leading to a reaction-type development with capabilities to handle biisotropic/nonactive/chiral/isotropic materials.**

I. Introduction

The basic Reciprocity Theorem of Electromagnetics was initially presented by Lorentz [1] for scalar fields, and generalized to vector fields, to become what we know today as the Lorentz form [2], [3]. Lorentz's contribution was a direct extension of the work of Rayleigh in vibrating mechanical systems [4] and optics [5]. Also, as pointed out in [6] and [7], it was Heaviside who, contemporary with the earliest work of Rayleigh, invoked what we know today as reciprocity for Electrical Networks.

Over the years, the basic theorem was extended to suit different circumstances, such as time harmonic fields, time domain, inhomogeneities, boundary conditions, anisotropy, piezoelectric media, and bianisotropy among others. Of special interest to us here is the work of Kong and Cheng [8], who considered the full bianisotropic case and departed from the previous line of thought. In order to state a reciprocity theorem in the vein of the Lorentz Theorem, they invoked a complementary space characterized by material different from that of the original space. This is important to us because biisotropy is a special case of bianisotropy, and in turn, chirality is a special case of biisotropy.

What this means is that a reciprocity theorem is available that applies to both chirality and biisotropy. Whereas chirality is inherently reciprocal (hence the term chiral reciprocal), biisotropy does require the introduction of a (different) complementary space. It is possible, however, to produce a new reciprocity relation for biisotropic and chiral materials by proper exploitation of the technique of decomposition of general fields into circularly polarized components [9], which has been used by this author [7] to obtain a nonLorentzian reciprocal relation for isotropic materials.

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The author is with Lockheed Martin, Princeton, NJ 08540 USA.
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The isotropic work [7] included applications and stressed the fact that the new theorem complements the old one in that the new one is eminently cross-polarized, while the old one is eminently co-polarized. As stated in [7], this concept is important, because it constitutes the foundation for a development of variational expressions that complements Rumsey's Reaction principle [10] and has potential to handle complex systems with high degree of cross-polarization.

The isotropic work in [7] resembles Tai's Complementary Theorem [11]; and it has been recently brought to the attention of the author[1] that one of the main results of [7] was apparently derived almost simultaneously and by entirely independent means by Fel'd and published in the Russian literature under a somewhat misleading title [12].

Aside from the notable work of Kong and Cheng on a reciprocity relation directly applicable to biisotropy, we can also cite the relevant works of Krowne [13] and Lindell *et al.* [14]. Here, we present a new reciprocity theorem that does not require the introduction of a complementary space.

II. Analysis

The constitutive relations for biisotropic media are [9]

$$\overline{D} = \varepsilon\overline{E} + \gamma\overline{H}, \quad \overline{B} = \mu\overline{H} + \beta\overline{E} \tag{1}$$

where the dimensions of γ and β are inverse to that of speed. The medium is lossless if E and μ are real, and $\gamma = \beta^*$. The condition for the medium to be reciprocal is $\gamma = -\beta$, and the resulting material is commonly known as chiral.

A general field decomposition in biisotropic media in terms of RCP/LCP (right/left circularly polarized) fields in the presence of electric $(\overline{J})$ and magnetic $(\overline{M})$ sources is possible via [9], [15]

$$\overline{E} = \overline{E}_+ + \overline{E}_- \quad \overline{H} = \overline{H}_+ + \overline{H}_- \tag{2}$$

$$\overline{J} = \overline{J}_+ + \overline{J}_- \quad \overline{M} = \overline{M}_+ + \overline{M}_- \tag{3}$$

$$\overline{E}_\pm(\overline{r}) = \mp j\eta_\pm\overline{H}_\pm(\overline{r}) \tag{4}$$

$$\eta_\pm = \sqrt{\frac{\mu}{\varepsilon} - \left(\frac{\gamma+\beta}{2\varepsilon}\right)^2} \mp j\left(\frac{\gamma+\beta}{2\varepsilon}\right) \tag{5}$$

$$k_\pm = \omega\left[\sqrt{\mu\varepsilon - \left(\frac{\gamma+\beta}{2}\right)^2} \pm j\left(\frac{\gamma-\beta}{2}\right)\right] \tag{6}$$

where $\eta_\pm/k_\pm$ refers to the wave impedance/number of the RCP/LCP field components. Note that the chiral case results in $\eta_+ = \eta_-$. On the other hand, $\gamma = \beta$ results in $k_+ = k_-$, defining a class of materials as broad as the chiral reciprocal, and referred to as the nonactive case [9].

Unlike chiral materials, which do not admit linearly polarized solutions, nonactive materials do allow linearly polarized fields, leading to very interesting effects such as magnetic dipole fields, which do not close, but which are open spiral lines that go from pole to pole [9], or Cherenkov radiation with helical magnetic field lines [9]. Fig. 1 illustrate these points.

Basic equations pertinent to the partial fields have been presented and solved in [9] and will not be repeated here since they will not be employed.

[1] Thanks are extended to A. Lakhtakia for reference to the paper of Fel'd.